THE HOMOGENEOUS BOLTZMANN-NORDHEIM EQUATION FOR BOSONS: LOCAL CAUCHY PROBLEM

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PDE seminar
Brown University, Providence, 24th October 2014
1. General presentation of the Boltzmann collisional model

2. A quantic version of the Boltzmann equation

3. Local existence and uniqueness of solutions

4. Further developments
GENERAL PRESENTATION OF THE BOLTZMANN COLLISIONAL MODEL
A probabilistic approach of systems of particles

- **System**: $N$ bodies moving in a domain $\Omega \subset \mathbb{R}^d$

- Particles are subject to external forces and interaction between particles

- Newton’s laws yield the reknown $N$-body problem (system of $2Nd$ equations)
A probabilistic approach of systems of particles

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- Newton’s laws yield the reknown $N$-body problem (system of $2Nd$ equations)

$\Rightarrow$ Already problematic when $N \geq 3$ !
A probabilistic approach of systems of particles

- **Idea**: focus on the average behaviour of the system
- establish the equation for the density function

\[ f : [0, T] \times \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}^+ \]

\[ (t, x, v) \mapsto f(t, x, v) \]

\[ f(t, x, v) dx dv \] is the probability of having a particle in \( B(x, dx) \) with a velocity in \( B(v, dv) \) at time \( t \)
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- \(f(t, x, v)dxdv\) is the probability of having a particle in \(B(x, dx)\) with a velocity in \(B(v, dv)\) at time \(t\)

\[
\Rightarrow \text{Minimal assumption : } \\
\forall t \in [0, T], \quad f(t, \cdot, \cdot) \in L^1_{loc} (\Omega, L^1_v (\mathbb{R}^d))
\]
The collisional process

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3. **Elastic collisions**:

   \[ v' + v_*' = v + v_* \]

   \[ |v'|^2 + |v_*'|^2 = |v|^2 + |v_*|^2 \]
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\]
4. **Microreversible process**: microscopic dynamics are reversible in time
The collisional process

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4. **Microreversible process**: microscopic dynamics are reversible in time

5. **Molecular chaos**: particles evolve independently
The Boltzmann Equation

\[ \forall t \geq 0, \forall (x, v) \in \Omega \times \mathbb{R}^d, \quad \partial_t f + v \cdot \nabla_x f = Q(f, f) \]
The Boltzmann Equation

**The collision operator:**

\[ Q(f, f) = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} |v - v_*|^\gamma b(\cos \theta) \left[ f' f'_* - ff_* \right] \, dv_* d\sigma \]

\[
\begin{align*}
  v' &= \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma \\
  v'_* &= \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma
\end{align*}
\]

, and \( \cos \theta = \langle \frac{v - v_*}{|v - v_*|}, \sigma \rangle \).
**The Boltzmann Equation**

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\]

, and \( \cos \theta = \langle \frac{v - v_*}{|v - v_*|}, \sigma \rangle \).

- \( \gamma \) belongs to \((-d, 1]\)
- \((b \circ \cos)\) is continuous in \((0, \pi]\), strictly positive near \(\theta \sim \pi/2\),

\[
b(\cos \theta) \sin^{d-2} \theta \sim b_0 \theta^{-(1+\nu)},
\]

for \(b_0 > 0\) and \(\nu\) in \((-\infty, 2)\)
Physical Observables

- **THE LOCAL DENSITY**: 
  \[ \rho(t, x) = \int_{\mathbb{R}^d} f(t, x, v) \, dv \]

- **THE LOCAL VELOCITY**: 
  \[ u(t, x) = \frac{1}{\rho(t, x)} \int_{\mathbb{R}^d} vf(t, x, v) \, dv \]

- **THE LOCAL ENERGY**: 
  \[ E(t, x) = \int_{\mathbb{R}^d} \frac{|v|^2}{2} f(t, x, v) \, dv = \rho(t, x) \frac{|u|^2}{2} + d \frac{\rho(t, x)\theta(t, x)}{2} \]
Conservation laws

- the preservation of the total mass

\[
\frac{d}{dt} \int_{\Omega} \rho(t, x) \, dx = 0,
\]
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- the preservation of total energy if \( \Omega \) has no boundary or if boundary conditions are bounce-back or specular reflections
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- the preservation of total energy if \( \Omega \) has no boundary or if boundary conditions are bounce-back or specular reflections
  \[ \frac{d}{dt} \int_{\Omega} E(t, x) \, dx = 0, \]

- the preservation of total momentum if \( \Omega \) has no boundary
  \[ \frac{d}{dt} \int_{\Omega} \rho(t, x) u(t, x) \, dx = 0, \]
H-Theorem and entropy dissipation

- **THE ENTROPY**: 
  \[ S(f) = \int_{\mathbb{R}^d} f \log f \, dx dv \]

- **THE ENTROPY DISSIPATION**: 
  \[ D(f) = -\int_{\mathbb{R}^d} Q(f, f) \log f \, dx dv \]

- **H-THEOREM**: 
  \[ \frac{d}{dt} S(f) = -\int_{\Omega} D(f) \, dx \leq 0. \]
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⇒ Irreversibility of the Boltzmann equation!
Local and global equilibria

- The H-theorem implies that local equilibria are **THE LOCAL MAXWELLIANS**:

\[
M(\rho(t,x),u(t,x),\theta(t,x))(v) = \frac{\rho}{(2\pi\theta)^{d/2}} e^{-\frac{|v-u|^2}{2\theta}}.
\]
The H-theorem implies that local equilibria are \textbf{THE LOCAL MAXWELLIANS}:

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\]

\textbf{GLOBAL EQUILIBRIA}:

\[
\forall (x, v) \in \Omega \times \mathbb{R}^d, \quad v \cdot \nabla_x M(\rho, u, \theta) = 0.
\]
Local and global equilibria

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  \]

- \textbf{Global equilibria}:
  \[
  \forall (x,v) \in \Omega \times \mathbb{R}^d, \quad v \cdot \nabla_x M(\rho,u,\theta) = 0.
  \]

- in the case of the torus or non-axis symmetric bounded domains with bounce-back or specular reflection boundary conditions, the only global equilibrium is
  \[
  \mu(v) = \frac{1}{(2\pi)^{d/2}} e^{-\frac{|v|^2}{2}}.
  \]
A QUANTIC VERSION OF THE BOLTZMANN EQUATION
Quantum effects in gases of bosons

- **Quantum collision properties**: The probability of two particles colliding also depends on the number of particles already in the outcoming velocity.
Quantum effects in gases of bosons

- **Quantic collision properties**: The probability of two particles colliding also depends on the number of particles already in the outcoming velocity.

- **The Boltzmann-Nordheim operator**:

\[
\int_{\mathbb{R}^N \times S^{d-1}} |v - v_*|^\gamma b(\cos \theta) \left[ f'(1 + f)f'(1 + f^*) - f(1 + f')f^*(1 + f^*) \right] dv_*
\]
Two types of equilibrium

- **Entropy of the solution**: always increasing in time

\[ S(f) = \int_{\mathbb{R}^d} [(1 + f)\log(1 + f) - f\log(f)] \, dv \]

\[ m_0 = 0 \Rightarrow \text{Unique for given } (M_0, v_0, T_0) \text{ and } m_0 = 0 \text{ iff } T_0 \geq T_c(M_0) \]
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⇒ No control over concentration phenomena
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⇒ No control over concentration phenomena

- **Equilibria**:

\[ F_{BE}(v) = m_0 \delta(v - v_0) + \frac{1}{e^{\frac{\beta}{2} (|v - v_0|^2 - \mu)}} - 1 \]

- \( m_0 \geq 0 \), \( \beta \) in \( (0, +\infty) \), \( -\infty < \mu \leq 0 \)
- \( \mu \cdot m_0 = 0 \)
Two types of equilibrium

- **Entropy of the solution**: always increasing in time

  \[ S(f) = \int_{\mathbb{R}^d} [(1 + f)\log(1 + f) - f\log(f)] \, dv \]

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Local existence and uniqueness of solutions
Our framework

- **Spatially homogeneous equation:**

  \[ \partial_t f = Q(f) \]

- solutions in \( L^\infty \cap L^1 \left( 1 + |v|^2 \right) \)
Our framework

- **Spatially homogeneous equation:**
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  - solutions in \( L^\infty \cap L^1 \left( 1 + |v|^2 \right) \)

- **The collision operator:**
  - Hard and Maxwellian potentials (\( 0 \leq \gamma \leq 1 \))
  - Grad’s angular cutoff (\( b \circ \cos \text{ integrable on the sphere} \))
Our framework

\[
\partial_t f = Q^+(f) - fQ^-(f)
\]

where we defined

\[
Q^+(f) = C_\Phi \int_{\mathbb{R}^N \times \mathbb{S}^{d-1}} |v - v_*|^\gamma b(\cos \theta) f' f'_*(1 + f + f_*) dv_* d\sigma
\]

\[
Q^-(f) = C_\Phi \int_{\mathbb{R}^N \times \mathbb{S}^{d-1}} |v - v_*|^\gamma b(\cos \theta) f_*(1 + f' + f'_*) dv_* d\sigma
\]
Previous studies

- **Isotropic Cauchy theory** : radially symmetric solutions
  - Lu (2000 – 2005) : global solutions in $L^1 \left(1 + |v|^2\right)$ and weak form for distributions
  - Escobedo-Velázquez (preprint) : locally in time in $L^\infty(1 + |v|^{6+0})$
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- **Bose-Einstein condensate**:
  - Lu: long-time convergence towards equilibrium
  - Escobedo-Velázquez: blow-up in finite time
Our Cauchy result

Theorem

Let \( f_0(v) \) be in \( L_{2,v}^{1} \cap L_{v}^{\infty} \).

Then there exists \( T_0 > 0 \) such that there exists a unique \( f \) in \( L_{\text{loc}}^{\infty} ([0, T_0), L_{2,v}^{1} \cap L_{v}^{\infty}) \) solution that preserves mass and energy. Moreover, this solution satisfies

- \( T_0 = +\infty \) or \( \lim_{T \to T_0^-} \| f \|_{L_{[0,T]}^{\infty} \times \mathbb{R}^d} = +\infty \),
- \( f \) preserves the momentum of \( f_0 \),
- for all \( s > 0 \) and for all \( 0 < T < T_0 \), the \( s^{th} \) moment is in \( L_{\text{loc}}^{\infty} ([T, T_0]) \).
Main arguments

- **From the Boltzmann equation**: we use the strategy of Mischler-Wennberg (1999)
  - Existence via an Euler scheme with a truncated $Q$
  - Uniqueness: quantification of the blow-up of the $(2 + \gamma)^{th}$-moment and Nagumo’s fixed point argument
Main arguments

- **From the Boltzmann equation**: we use the strategy of Mischler-Wennberg (1999)
  - Existence *via* an Euler scheme with a truncated $Q$
  - Uniqueness: quantification of the blow-up of the $(2 + \gamma)^{th}$-moment and Nagumo’s fixed point argument

- **Improvements**:
  - A new Povzner-type inequality (control of evolution of convex/concave functions through a collision)
  - New $L^\infty$ control on the gain operator *via* the Carleman representation
A priori estimates

- preservation of mass and energy
A priori estimates

- Preservation of mass and energy

**Difficulty for the $L^\infty_v$-norm:**

- $Q^-(f)(v) \geq C_1 (1 + |v|^\gamma) \|f\|_{L^1_v} - C_2 \|f\|_{L^2_v}$

\[
\|Q^+(f)\|_{L^\infty_v} \leq C_3 \|f\|_{L^\infty_v} \left[ C(\lambda) \|(1 + |v|^\gamma) f\|_{L^\infty_v} + \frac{2^{d-2}}{\lambda^{d-1}} \|f\|_{L^1_v} \right]
\]
A priori estimates

- Preservation of mass and energy

- **Difficulty for the $L_v^\infty$-norm:**
  - $Q^-(f)(v) \geq C_1 (1 + |v|^{\gamma}) \|f\|_{L_v^1} - C_2 \|f\|_{L_{2,v}^1}$

\[
\|Q^+(f)\|_{L_v^\infty} \leq C_3 \|f\|_{L_v^\infty} \left[ C(\lambda) \| (1 + |v|^{\gamma}) f \|_{L_v^\infty} + \frac{2^{d-2}}{\lambda d-1} \|f\|_{L_{2,v}^1} \right]
\]

- **Where to look:** if $f$ is solution in $L_{loc}^{\infty}([0, T_0), L_{2,v}^1 \cap L_v^\infty)$
  then for all $0 \leq T < T_0$, there exists $C_T > 0$,
  
  \[
  \sup_{[0, T] \times \mathbb{R}^d} \left( f(t, v) + \int_0^t (1 + |v|^{\gamma}) f(s, v) \, ds \right) \leq C_T.
  \]
The explicit Euler scheme

- **Truncated operator**: we consider $Q_n$ where the kernel is 
  
  \[ (n \wedge |v - v_*|)^\gamma \]
The explicit Euler scheme

- **Truncated operator**: we consider $Q_n$ where the kernel is $(n \wedge |v - v_*|)\gamma$

- **Combination with a Euler scheme**: solving

  $$\partial_t f_n = Q_n(f_n)$$

  with a Euler scheme on $[0, T_0]$ with a timestep $\Delta_n$:

  $$\begin{cases}
  f_n^{(0)}(v) = f_0(v) \\
  f_n^{(k+1)}(v) = f_n^{(k)}(v) \left(1 - \Delta_n Q_n^{-} \left(f_n^{(k)}(v)\right)\right) + \Delta_n Q_n^{+} \left(f_n^{(k)}(v)\right)
  \end{cases}$$
The explicit Euler scheme

- **Truncated operator**: we consider \( Q_n \) where the kernel is \( (n \wedge |v - v_\star|)^\gamma \)

- **Combination with a Euler scheme**: solving

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\]

with a Euler scheme on a \([0, T_0]\) with a timestep \( \Delta_n \) :

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\begin{cases}
  f_n^{(0)}(v) = f_0(v) \\
  f_n^{(k+1)}(v) = f_n^{(k)}(v) \left( 1 - \Delta_n Q_n^{-}(f_n^{(k)}) \right) + \Delta_n Q_n^{+}(f_n^{(k)})
\end{cases}
\]

- **Global in time?**: time \( T_0 \) depends on \( \|f_0\|_{L_v^\infty} \) and at \( T_0 \) we only have \( \|f_\infty\|_{L_v^\infty} \leq 2 \|f_0\|_{L_v^\infty} \)
The necessity of controlling moments
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- **Evolution of** $\| f - g \|_{L^1_v}$: for all $0 \leq t \leq T < T_0$,

\[
\frac{d}{dt} \| f - g \|_{L^1_v} \leq C_T \left[ \| f - g \|_{L^1_v} + \| f - g \|_{L^2_v} \right]
\]
The necessity of controlling moments

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- **Evolution of** $\| f - g \|_{L_2^v}$: for all $0 \leq t \leq T < T_0$,

$$\frac{d}{dt} \| f - g \|_{L_2^v} \leq C_T \left[ M_{2+\gamma}(t) \| f - g \|_{L_1^v} + \| f - g \|_{L_2^v} + \| f - g \|_{L_\infty} \right]$$
The necessity of controlling moments

- **Evolution of** $\|f - g\|_{L^1_v}$: for all $0 \leq t \leq T < T_0$,
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  \]

- **Evolution of** $\|f - g\|_{L^1_{2,v}}$: for all $0 \leq t \leq T < T_0$,
  \[
  \frac{d}{dt} \|f - g\|_{L^1_{2,v}} \leq C_T \left[ M_{2+\gamma}(t) \|f - g\|_{L^1_v} + \|f - g\|_{L^2_v} + \|f - g\|_{L^\infty_{[0,T],v}} \right]
  \]

- **Control of** $\|f - g\|_{L^\infty_{[0,T],v}}$: there exists $\tau < T_0$, for all $0 \leq t \leq \tau$,
  \[
  \|f - g\|_{L^\infty_{[0,T],v}} \leq C_T \sup_{[0,t],v} \|f - g\|_{L^1_{2,v}}
  \]
Existence and blow-up of moments

**Evolution of quantities through collision:**

\[
\int_{\mathbb{R}^d} Q(f) \psi \, dv = \int_{\mathbb{R}^d \times \mathbb{R}^d \times S^{d-1}} q(f) \left[ \psi_*' + \psi' - \psi_* - \psi \right] \, d\sigma \, dv \, dv_*
\]

with

\[
q(f)(v, v_*) = |v - v_*|\gamma b(\cos \theta) ff_* (1 + f' + f_*')
\]
Existence and blow-up of moments

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with

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q(f)(v, v_*) = |v - v_*|^\gamma b(\cos \theta) \, ff_* \left(1 + f' + f'_*\right)
\]

- **Povzner-type inequality:** as an example for

\[1 \leq F \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d \times S^{d-1})\]
Existence and blow up of moments

\[
\int_{S^{d-1}} F(v, v_*, \sigma) b(\theta) \left( \psi(|v'|^2) + \psi(|v'|^2) - \psi(|v_*|^2) - \psi(|v|^2) \right) d\sigma
\]

\[
= G_\psi(v, v_*) - H_\psi(v, v_*)
\]

And we have, if \( \psi(x) = x^{1+\alpha} \) with \( \alpha > 0 \),

1. \( |G(v, v_*)| \leq C_G \alpha (|v| |v_*|)^{1+\alpha} \)

2. \( H(v, v_*) \geq C_H \alpha \left( |v|^{2+2\alpha} + |v_*|^{2+2\alpha} \right) \left[ 1 - 1_{\{|v| < |v_*| < 2|v|\}} \right] \)
Existence and blow up of moments

- **Creation of moments**: For \( s > 0 \), as soon as \( T > 0 \),

\[
\int_{\mathbb{R}^d} |v|^s f(t, v) \, dv \in L^\infty_{\text{loc}} ([T, T_0])
\]
Existence and blow up of moments

- **Creation of moments**: For $s > 0$, as soon as $T > 0$,

  \[
  \int_{\mathbb{R}^d} |v|^s f(t, v) \, dv \in L_{\text{loc}}^\infty ([T, T_0])
  \]

- **Behaviour at $t = 0$**: there exists $0 < \tau < T_0$ and there exists $C_{\tau} > 0$ such that

  \[
  \forall t \in (0, \tau], \quad M_{2+\gamma}(t) \leq \frac{C_{\tau}}{t}
  \]
For all $0 < t \leq T < T_0$,

$$\frac{d}{dt} \| f - g \|_{L^1_v} \leq C_T \left[ \| f - g \|_{L^1_v} + \| f - g \|_{L^1_{2,v}} \right]$$

$$\frac{d}{dt} \| f - g \|_{L^1_{2,v}} \leq C_T \left[ M_{2+\gamma}(t) \| f - g \|_{L^1_v} + \| f - g \|_{L^1_{2,v}} + \| f - g \|_{L^\infty_{[0,T],v}} \right]$$

$$\| f - g \|_{L^\infty_{[0,t],v}} \leq C_T \sup_{[0,t],v} \| f - g \|_{L^1_{2,v}}$$
Nagumo’s fixed point argument

For all $0 < t \leq T < T_0$,

\[
\begin{align*}
\|f - g\|_{L^1_v} &\leq C_T t \\
\frac{d}{dt} \|f - g\|_{L^1_{2,v}} &\leq C_T \left[ \frac{1}{t} \|f - g\|_{L^1_v} + \|f - g\|_{L^1_{2,v}} + \|f - g\|_{L^\infty_{[0,T],v}} \right] \\
\|f - g\|_{L^\infty_{[0,T],v}} &\leq C_T \sup_{[0,t],v} \|f - g\|_{L^1_{2,v}}
\end{align*}
\]
Nagumo’s fixed point argument

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\|f - g\|_{L^1_v} \leq C_T t \\
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Nagumo’s fixed point argument

For all \(0 < t \leq T < T_0\),

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\]

\[
\frac{d}{dt} \| f - g \|_{L^1_{2,V}} \leq C_T \left[ M_{2+\gamma}(t) \| f - g \|_{L^1_V} + \| f - g \|_{L^1_{2,V}} + \| f - g \|_{L^\infty_{[0,T],V}} \right]
\]

\[
\| f - g \|_{L^\infty_{[0,t],V}} \leq C_T t
\]
For all $0 < t \leq T < T_0$, 

\[ \|f - g\|_{L^1_v} \leq C_T t^2 \]

\[ \frac{d}{dt} \|f - g\|_{L^1_{2,v}} \leq C_T \left[ M_{2+\gamma}(t) \|f - g\|_{L^1_v} + \|f - g\|_{L^1_{2,v}} + \|f - g\|_{L^\infty_{[0,T],v}} \right] \]

\[ \|f - g\|_{L^\infty_{[0,t],v}} \leq C_T t \]
Nagumo’s fixed point argument

For all $0 < t \leq T < T_0$,

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\]
\[
\|f - g\|_{L^\infty_{[0,t],v}} \leq C_T t^n
\]
Nagumo’s fixed point argument

For all \(0 < t \leq T < T_0\),

\[
\|f - g\|_{L^1_v} \leq C_T t^n \\
\|f - g\|_{L^{1}_{2,v}} \leq C_T t^n \\
\|f - g\|_{L^\infty_{[0,t],v}} \leq C_T t^n
\]

\(\Rightarrow\) \(C_T\) depends on \(n\)!
Nagumo’s fixed point argument: the conclusion

\[ \forall t \in [0, T], \quad \frac{d}{dt} \| f - g \|_{L^1_{2,v}} \leq \frac{K_1}{t} \| f - g \|_{L^1_{2,v}} + K_2 \sup_{[0,t],v} \| f - g \|_{L^1_{2,v}} \]

1. Look at \( X(t) = \| f - g \|_{L^1_{2,v}} / t^n \): \( X(0) = 0, X'(0) = 0 \)
Nagumo’s fixed point argument : the conclusion

∀t ∈ [0, T], \[ \frac{d}{dt} \| f - g \|_{L^1_{2,ν}} \leq \frac{K_1}{t} \| f - g \|_{L^1_{2,ν}} + K_2 \sup_{[0,t],ν} \| f - g \|_{L^1_{2,ν}} \]

1. Look at \( X(t) = \| f - g \|_{L^1_{2,ν}} / t^n : X(0) = 0, X'(0) = 0 \)
2. Choose \( n \) large enough such that
\[ X(t) \leq tK_2 \sup_{[0,t],ν} X(s) \]
Nagumo’s fixed point argument : the conclusion

\[ \forall t \in [0, T], \quad \frac{d}{dt} \| f - g \|_{L^1_{2,v}} \leq \frac{K_1}{t} \| f - g \|_{L^1_{2,v}} + K_2 \sup_{[0,t],v} \| f - g \|_{L^1_{2,v}} \]

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   \[ X(t) \leq tK_2 \sup_{[0,t],v} X(s) \]
3. By induction
   \[ X(t) \leq (tK_2)^m \sup_{[0,t],v} X(s) \]
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4. On \([0, \tau]\), we have \( X(t) = 0 \). Spread on \([\tau', T]\) by Gronwall’s Lemma
In an ideal future:

- Prove the creation of the Bose-Einstein condensate in a non-isotropic setting (we do not even know criterion for global existence...)
- Obtain a constructive proof of the creation of the condensate (we do not even know if it is unique...)
1 **In an ideal future:**

- Prove the creation of the Bose-Einstein condensate in a non-isotropic setting (we do not even know criterion for global existence...)
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2 **In the real world:**

- Understand how to obtain global in time solutions
- Understand if non-equilibrium blow-ups can occur
- Understand the collisional process leading to condensate for non-isotropic setting