A Return Journey From Boltzmann To Navier-Stokes Equations

March 21, 2012
Aim of the Projet

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- We want to study the Boltzmann Equation on the torus as the number of collisions between particles tends to infinity.
- We want to understand the link between Boltzmann equation and Incompressible Navier-Stokes equation.
- We want to have a flavour of a possible reverse way from fluid dynamics to Boltzmann equation.
1. Boltzmann equation and fluid dynamics

2. A link between the two approaches

3. From Boltzmann to Navier-Stokes

4. From Navier-Stokes to Boltzmann: two points of view
\( (t, x, v) \in \mathbb{R}^+ \times T^N \times \mathbb{R}^N: \)

\[
\partial_t f + v \cdot \nabla_x f = \frac{1}{\varepsilon} Q(f, f) = \frac{1}{\varepsilon} \int_{\mathbb{R}^N \times S^{N-1}} B(|v - v_*|, \cos \theta) \left[ f' f'_* - f f_* \right] dv_* d\sigma
\]

**Notations:**

\[
\begin{align*}
  v' &= \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma \\
  v_*' &= \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma \\
  \theta &= \text{angle}(v, v_*, \sigma)
\end{align*}
\]
- **Direct Observations:**
  - $f$ represents the density of probability of position and velocity of particles at $t$
  - $Q$ is a bilinear operator acting only on the $v$ variable
  - the equation has a global Maxwellian equilibrium:

$$M = M(v) = \frac{\rho}{(2\pi T)^{n/2}} e^{-\frac{|v-u|^2}{2T}}$$
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$\Rightarrow$ Hydrodynamical Limit: what happen to $f$ as $\varepsilon \to \infty$?
Several Models:

- Systems of equations linking $\rho$, $u$ and $\theta$
- On short times: Acoustic Equations
- On greater time scale: Euler, Stokes, Navier-Stokes Equations
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Link with Collisional Models:
- Kinetic: describes movements of particles in rarefied gases (Boltzmann Equation)
- Fluids: describes macroscopic moves of particles in contact
- As $\varepsilon$ decreases we guess that Kinetic Models will converge to Fluid Models
A Link Between the Two Approaches
Macroscopic quantities associated to a density $f_\epsilon$:

- particles density

$$\rho(x) = \int_{\mathbb{R}^N} f_\epsilon(x, v) dv,$$

- mean velocity

$$u(x) = \frac{1}{\rho(x)} \int_{\mathbb{R}^N} vf_\epsilon(x, v) dv,$$

- temperature

$$\theta(x) = \frac{1}{N} \int_{\mathbb{R}^N} (|v|^2 - N) f_\epsilon(x, v) dv.$$
Rescaling in time:

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Perturbation around the equilibrium:

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f_\varepsilon = M + \delta_\varepsilon \sqrt{M} h_\varepsilon
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The Perturbative Equation:

$$\partial_t h_{\varepsilon} + \frac{1}{\tau_{\varepsilon}} v \cdot \nabla x h_{\varepsilon} = \frac{1}{\varepsilon \tau_{\varepsilon}} L(h_{\varepsilon}) + \frac{\delta_{\varepsilon}}{\varepsilon \tau_{\varepsilon}} \Gamma(h_{\varepsilon}, h_{\varepsilon})$$
Convergences of models in a weak sense:

- If \( \tau_\varepsilon = \delta_\varepsilon \) and \( \lim_{\varepsilon \to 0} \frac{\delta_\varepsilon}{\varepsilon} = 0 \) convergence to Incompressible Euler equations.
- If \( \tau_\varepsilon = \delta_\varepsilon = \varepsilon \) convergence to Incompressible Navier-Stokes equations.
- If \( \tau_\varepsilon = \varepsilon \) and \( \lim_{\varepsilon \to 0} \frac{\delta_\varepsilon}{\varepsilon} = 0 \) convergence to Incompressible Stokes equations.

⇒ We focus on the Navier-Stokes case:

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\partial_t h_\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla h_\varepsilon = \frac{1}{\varepsilon^2} L(h_\varepsilon) + \frac{1}{\varepsilon} \Gamma(h_\varepsilon, h_\varepsilon)
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$\Rightarrow$ We focus on the Navier-Stokes case:

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From Boltzmann to Navier-Stokes
There exists $0 < \varepsilon_N \leq 1$ and $k_0 \in \mathbb{N}$ such that for any $k \geq k_0$,

1. there exists a norm $\| \cdot \|_{\mathcal{H}_\varepsilon^k}$, such that for all $0 < \varepsilon \leq \varepsilon_N$:

$$\| \cdot \|_{\mathcal{H}_\varepsilon^k}^2 \sim \| \cdot \|_{L_{x,v}^2}^2 + \sum_{|l| \leq k} \| \partial_l^0 \cdot \|_{L_{x,v}^2}^2 + \varepsilon^2 \sum_{|l| + |j| \leq k} \| \partial_j^l \cdot \|_{L_{x,v}^2}^2,$$

2. For $f_{in} = M + \varepsilon M^{1/2} h_{in}$ such that

$$\| h_{in} \|_{\mathcal{H}_\varepsilon^k} \leq \delta_k,$$

there exists a unique $0 \leq f_\varepsilon = f_\infty + \varepsilon f_\infty^{1/2} h_\varepsilon$ in $C(\mathbb{R}^+ , H_x^k)$ such that

$$\| h_\varepsilon \|_{\mathcal{H}_\varepsilon^k} \leq \delta_k e^{-\tau_k t}.$$
• Comments and remarks:
  • The exponential decay can be obtained even for the $v$-derivatives
  • $(h_\varepsilon)$ is uniformly bounded in $L^\infty_t H^k_x L^2_v$
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Weak convergence:

- \( (h_\varepsilon) \) converges weakly-* to \( h \) in \( L^\infty_t H^k_x L^2_v \) such that

\[
h(t, x, v) = \left[ \rho(t, x) + v \cdot u(t, x) + \frac{1}{2} (|v|^2 - N) \theta(t, x) \right] M(v)^{1/2},
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Comments and remarks:

- The exponential decay can be obtained even for the \( v \)-derivatives
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- \( (\rho, u, \theta) \) satisfies the Incompressible Navier-Stokes equations together with the Boussinesq relation

\[
\nabla_x (\rho + \theta) = 0.
\]
Theorem

- We have that \( \text{div}(u) = 0 \) and \( \rho + \theta = 0 \).
- Moreover, \( \int_0^T h dt \) belongs to \( H^k_x L^2_v \) and it exists \( C > 0 \) such that for all \( T > 0 \),

\[
\left\| \int_0^T h dt - \int_0^T h \epsilon dt \right\|_{H^k_x L^2_v} \leq C \max\{\sqrt{\epsilon}, T^{3/2} \sqrt{\epsilon}\}.
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- Initial conditions for Navier-Stokes equations:
  - \( u(0, x) = P u_{in}(x) \), the divergence-free part of \( u_{in}(x) \),
  - \( \rho(0, x) = -\theta(0, x) = \frac{1}{2} (\rho_{in}(x) - \theta_{in}(x)) \).
FROM NAVIER-STOKES TO BOLTZMANN: TWO POINTS OF VIEW
\[ \begin{align*} 
\partial_t u - \nu \Delta u + u \cdot \nabla u + \nabla p &= 0, \\
\nabla \cdot u &= 0, \\
\partial_t \theta - \kappa \Delta \theta + u \cdot \nabla \theta &= 0, 
\end{align*} \]
Plan

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Conclusion

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- Approximating a solution of Incompressible Navier-Stokes equations:
  finding a global Maxwellienne \( M(\rho^*,u^*,\theta^*) \) to obtain expected \( \nu \) and \( \kappa \)
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- **Approximating a solution of Incompressible Navier-Stokes equations:**
  finding a global Maxwellienne \( M(\rho^*, u^*, \theta^*) \) to obtain expected \( \nu \) and \( \kappa \)

- **Constructing solutions near the fluid:**
  constructing a solution to Boltzmann equation near local Maxwellienne \( M(\rho, u, \theta) \)
Thank you for your attention